February 2016 IJESR Volume 4, Issue 2 ISSN: 2347-6532

SOME RESULTS ON SUB-RIEMANNIAN GEOMETRY

Sarika M. Patil*

T. Venkatesh**

Abstract:

Sub- Riemannian structures naturally occur in different branches of Mathematics in the study of constrained systems in classical mechanics, in optimal control, geometric measure theory and differential geometry. In this paper, we show that Sub- Riemannian structures on three manifolds locally depend on two functions ϕ_1 and K of three variables and we investigate how these differential invariants influence the geometry.

Keywords: Sub-Riemannian Structure, K_0-structures, Homogeneous Manifolds.

2000 AMS Mathematics Subject Classification: 53B20, 53B35

* Department of Mathematics, Government First Grade College, Hubli, Karnataka State, India,

** Department of Mathematics, Rani Channamma University, Belgaum, Karnataka State, India

1. Introduction

Sub-Riemannian structures naturally occur in different branches of Mathematics in the study of constrained systems in classical mechanics, in optimal control, geometric measure theory and differential geometry. In the first section, we deals with some basic definition of the concepts and in the second section, we solve the local equivalence problem for sub-Riemannian metrics on three manifolds. We obtain the differential invariants distinguishing sub-Riemannian structures and we interpret these invariants geometrically. In the third section, deals with the homogeneous manifolds and related examples.

Definition 1.1: A topological M is a manifolds of dim n if

(i) M is Housdorff

(ii) M is Second Countable and

(iii) M is Locally Euclidean of dim n.

Definition 1.2: A Co-ordinate chart on M is a pair (U, ϕ), where U $\subseteq$ M is open and ϕ: U $\rightarrow$ ϕ(U) $\subseteq \mathbb{R}^{\alpha}$ is a homomorphism. The set U is called a co-ordinate domain or co-ordinate neighbourhood or co-ordinate patch. If ϕ(U) is a ball in $\mathbb{R}^{\alpha}$, U is called co-ordinate ball. A co-ordinate chart (U, ϕ) is centered at p if $\phi(p) = 0$.

From the Darboux Theorem, we know that every two contact manifolds of the same are locally equivalent, Sub-Riemannian structures already have local invariants has been shown in [2], when n = 1 there are functions K and the Eigen value of the torsion matrix which do not change under the local automorphism of a Sub-Riemannian structures on 3- manifolds were defined in [1].

In [6] and [8] similar invariants were considered and further generalized to the case of contact metric manifolds in [7]. Note that all these invariants coincide when the dimension of a manifold is three.

A sympletic manifold (M, ω) is a smooth manifold M together with a smooth, closed, non-degenerate 2-form ω called the sympletic form. Note that the non-degeneracy condition on ω requires that M is even –dimensional. A Hamiltonian on M is a smooth function H: M $\rightarrow \mathbb{R}$ to every Hamiltonian H on M there is a associated a vector field ξ_H on M, defined by the condition $dH(V) = \omega(V, \xi_H), \forall\, V \in TM$.

2. Sub Riemannian K_0 Structures on Manifolds

We define the K- structures that completely characterizes the Sub Riemannian structures.

Given an n-dimensional manifold M, every local coframing $\eta = \{\eta^1, \eta^2, \ldots \eta^n\}$ on $U \subset M$ determines a sub- Riemannian structure (D,<, >) on U by setting $D = \{\eta^1, \eta^2, \ldots \eta^n\}^{\perp}$ and $<,> = (\eta^1)^2, (\eta^2)^2, \ldots\ldots\ldots. (\eta^m)^2 /D$.

Conversely, given a Sub-Riemannian structures (D, < , >) on U, we can always choose a local coframing η that satisfies $D = \{\eta^{m+1}, \ldots\ldots \eta^n\}^{\perp}$ and $< , > = (\eta^1)^2, (\eta^2)^2, \ldots\ldots\ldots. (\eta^m)^2 /D$. Now such a choice of coframing η is not unique for D determines $\eta^1, \eta^2, \ldots \eta^n$ only upto a $Gl(n - m, \mathbb{R})$ action, the quadratic form $(\eta^1)^2 + (\eta^2)^2 + \cdots + (\eta^m)^2$ determines $\eta^1, \eta^2, \ldots \eta^m$ only upto an O(m) action and, furthermore, since $<,> = (\eta^1)^2, (\eta^2)^2, \ldots\ldots\ldots (\eta^m)^2 /D$, we may add arbitrary multiples of the forms $\eta^{m+1}, \ldots\ldots, \eta^n$ to each η^i for $1 \le i \le m$.

Let us say a coframing $\eta = (\eta^1, \eta^2, \ldots\ldots \eta^n)$ is O-adopted coframes of (M, D, <, >) forms a K_0-structure $B_0 \to M$, where the structure group is

$$K_0 = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} : A \in O(m), B \in M(m, n\text{-}m), C \in Gl(n\text{-}m, R) \right\}.$$

Now specifying the sub-Riemannian structure (D, < , >) on M is equivalent to specifying the K_0-structure. In particular sub-Riemannian structures are locally equivalent if and only if their corresponding K_0-structures are locally equivalent.

The sub-Riemannian manifold (M, D, < , >) is said to be complete if M is complete with respect to the metric d. The next proposition gives useful equivalent criterion for completeness. Its proof follows from general topology and will be omitted.

Proposition 2.1: The following are equivalent:

(i) (M, D, < , >) is complete.

(ii) The closed balls are compact.

(iii) There is a nested sequence $\{K_n\}$ of compact sets with $M = \bigcup K_n$ such that if $q_n \notin K_n$ then $d(P, q_n) \to \infty$.

(iv) Every D - curve that leaves every compact set has definite length.

Proposition 2.2: There exists a complete sub-Riemannian manifolds that has a non-complete associated Riemannian metric.

Proof: Let us consider M = $\mathbb{R}^3$ with the sub-Riemannian structure defined by

$$D=\ker\left(dz - \frac{1}{2} r^2 d\theta\right),$$

$$<.> = \frac{1}{1+z^2}(dr^2 + r^2 d\theta^2),$$

where (r, θ, z) are cylindrical co-ordinates on $\mathbb{R}^3$. Set $\omega = dz - \frac{1}{2} r^2 d\theta$.

Observe that $\omega \wedge d\omega = - rdz \wedge dr \wedge d\theta \neq 0$; therefore D is a contact distribution. The associated Riemannian metric $< , > + \omega^2$ is not complete. Let γ_0 be the curve described by (r, θ, z) = (0, 0, t). Its length in the metric $< , > + \omega^2$ is equal to the integral over the positive t-axis of $\frac{1}{1+t^2}$. Therefore γ_0 is a curve that leaves every compact set but has finite length.

On the other hand, we claim that < , > is complete. First observe that if $\gamma(t) = (r(t), \theta(t), z(t))$ is a D - curve that leaves every compact set but stays bounded in z, i.e. $|z(t)| \le z_0$, then the length of γ is

$$L(\gamma) = \lim_{h\to\infty} \int_0^h \frac{1}{\sqrt{1+z^2(t)}} \sqrt{\dot{r}^2(t) + r^2(t)\,\dot{\theta}^2(t)}\,dt$$

$$\ge \frac{1}{\sqrt{1+z_0^2}} \lim_{h\to\infty} \int_0^h |\dot{r}(t)|\,dt$$

$$\ge \frac{1}{\sqrt{1+z_0^2}} \lim_{h\to\infty} (r(h) - r(0)).$$

Because γ leaves every compact set, $\lim_{h\to\infty} r(h) = \infty$, and therefore γ has infinite length.

To show that the D– curves that escape every z-bound have infinite length, we will find another associated Riemannian metric $< , >_R$ for which $< , >_R \ge dz^2$.

Set $<\ ,\ >_R = <\ ,\ > + \frac{\omega}{1+z^2}(\mu_1 dr + \mu_2 d\theta + (1+z^2)dz)$, where μ_1 and μ_2 are continuous functions.

We can easily verify that the quadratic form $<\ ,\ >_R - dz^2$ is non –negative.

Therefore, every curve that escapes every z-bound has infinite length with respect to $<\ ,\ >_R$ and thus (M, D, < , >) is complete.

3. Reduction of the K_0-Structure for the 3-dimensional Case:

For sub-Riemannian three-manifolds, we show how reduce the structure group to the group O(2). As a consequence, we see that two sun-Riemannian structures on a three-manifold M^3 are equivalent if and only if a canonical cofrmaing on a certain S^1-bundle $B_2 \to M^3$ is preserved.

If (M, D, < , >) is a 3-dimensional sub – Riemannian manifold, then the bracket generating distribution D is necessarily a contact distribution, and so every O-adopted coframing $\eta = (\eta^1, \eta^2, \eta^3)$ satisfies $\eta^3 \wedge d\eta^3 \neq 0$. The structure group for the O-adopted coframes is

$$G_0 = \left\{ \begin{bmatrix} A & b \\ 0 & c \end{bmatrix} : A \in O(2),\ b \in \mathrm{R}^2, c \in \mathrm{R} - \{0\} \right\},$$

and the structure equations $d\omega = -\theta \wedge \omega + T\omega \wedge \omega$ are in this case.

We can modify, β, γ and δ so as to absorb the first two columns of T, and we can modify α so as to absorb the top two entries T^1_{12} and T^2_{12} of the last column of T , T^3_{12} cannot possibly be absorbed since D is a contact distribution. Thus the structure equations can be written as

$$d\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = -\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ 0 & 0 & \delta \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T^3_{12} \end{pmatrix} \begin{pmatrix} \omega^2 \wedge \omega^3 \\ \omega^3 \wedge \omega^1 \\ \omega^1 \wedge \omega^2 \end{pmatrix}$$

The choice of pseudo – connection $\theta = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ 0 & 0 & \delta \end{pmatrix}$

for which the structure equations are of the above form is not unique, for we may add arbitrary multiples of ω^3 to β, γ and δ.

This equation $R_g^*\omega = g^{-1}\omega$ implies $R_g^*\omega^3 = c^{-1}\omega^3$, and taking the exterior derivatives of both sides gives.

Thus the structure equation can be written as

$$d\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = -\begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & -a_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^2 \wedge \omega^3 \\ \omega^3 \wedge \omega^1 \\ \omega^1 \wedge \omega^2 \end{pmatrix} \quad \text{---- (i)}$$

where we have set $a_1 = T_{23}^1 = -T_{31}^2$ and $a_2 = T_{31}^1 = -T_{23}^2$. Now the choice of pseudo-connection is unique, for $\theta \wedge \omega = 0$ if and only if $\alpha = 0$. By differentiating the structure equation (i), we get

$$da_1 = -2a_2\alpha + \sum_{i=1}^{3} A_i\omega^i \qquad ---(ii)$$

$$da_2 = 2a_1\alpha + \sum_{i=1}^{3} B_i\omega^i \qquad ---(iii)$$

$$d\alpha = (A_1 - B_1)\,\omega^2 \wedge \omega^3 + (A_1 + B_2)\,\omega^3 \wedge \omega^1 + K\,\omega^1 \wedge \omega^2 \qquad \text{---- (iv)}$$

Proposition 3.1: The functions $a_1^2 + a_2^2$ and K_0 are well defined on M^3.

Proof: By equations (ii) and (iii) to compute that the exterior derivative of $a_1^2 + a_2^2$ is equal to zero modulo the semi-basic one forms ω^i, from which it follows that $a_1^2 + a_2^2$ is well defined on M^3. Similarly, if we differentiate both sides of equation (iv) and then sedge with ω^3, we find that dK_0 is equal to zero modulo ω^i.

Theorem 3.2: Let (M^3, D, < , >) be a sub-Riemannian manifold, and let [p] be a point in the leaf space N. As p moves along the fiber $\phi_N^{-1}[p]$ with unit speed the corresponding inner product $(<,>_N)_p$ on $T_{[p]}N$ defines a curve in $S_{[p]}$ whose speed is $2\sqrt{a_1^2 + a_2^2}$.

Proof: Choose coordinates as above so that [p] = (0, 0). Now the fiber is the set {(0, 0, z) / z∈ℝ} and the metric space ω^3 on the fiber is just dz. The curve in $S_{[p]}$ corresponding to the unit speed curve γ(t) = (0, 0, t) in the fiber is

$$t \to \begin{bmatrix} E(0,0,t) & F(0,0,t) \\ F(0,0,t) & G(0,0,t) \end{bmatrix}$$

The squared speed of this curve with respect to the Poincare metric is equal to $F_z^2\text{-}E_zG_z$.

The Lie derivative $(\omega^1)^2 + (\omega^2)^2$ in the V direction is easily seen to be equal to $a_2(\omega^1)^2\text{-}a_2(\omega^2)^2\text{-}2a_1\omega^1\omega^2$.

On the other hand, in adopted coordinates, the Lie derivative in the direction $V = \partial/\partial z$ is equal to $E_z dx^2 + 2F_z\, dxdy + G_z dy^2$. Because the area form $\omega^1 \wedge \omega^2$ is equal to $dx \wedge dy$, it follows that

$$F_z^2\text{-}E_zG_z = (-2a_1)^2 - (2a_2)(-2a_2) = 4(a_1^2 + a_2^2)$$

and therefore the speed of the curve is equal to $2\sqrt{a_1^2 + a_2^2}$.

Proposition 3.3: Let (M^3, D, < , >) be a amenable. The induced inner product $(<\,,>_N)_p$ on $T_{[p]}N$ gives a well defined metric on N if and only if $a_1^2 + a_2^2 = 0$. In this case, the Gaussian Curvature of N is equal to K_0.

Proof: The first statement follows from theorem 3.3.2. To compute the curvature of N with the metric $<\,,>_N$, we choose a local section $\sigma: W \rightarrow M^3$ for some suitably small neighbourhood $W \subseteq N$, and consider the pull back bundle $\sigma^{-1}B_2$ over W. If $\eta = (\eta^1, \eta^2, \eta^3)$ is any 2-adopted coframing then $(\sigma^*\eta^1, \sigma^*\eta^2)$ is an orthonormal coframing of W. Because η^3 is uniquely determined, the bundle F_0 of orthonormal coframes for W can be identified with $\sigma^{-1}B_2$.

With this identification, we let $(\sigma, 1)$ be the map from F_0 to B_2 that sends a point $([p], \eta_p)$ in F_0 to the point $(\sigma[p], \eta_p)$ in $\sigma^{-1}B_2$.

Unwinding definitions, it is clear that the pull – backs $(\sigma, 1)^*\omega^1$ and $(\sigma, 1)^*\omega^2$ are the tautological one – forms on F_0. Now imply that $(\sigma, 1)^*\alpha$ is the connection form on F_0 and therefore $(\sigma, 1)^*K_0 = \sigma^*K_0$ is the Gaussian curvature of N.

4. The Homogeneous Manifolds:

The homogeneous sub-Riemannian manifolds are those for which the group automorphisms of the reduced K-structure acts transitively.

Theorem 4.1: The homogeneous manifolds can be classified as follows:

1) $\lambda_1 = \lambda_2 = 0$
2) $\lambda_2 = 0$ and $\lambda_3 = \phi_1$
3) $\lambda_1 = 0$ and $\lambda_3 = -\phi_1$, where ϕ_1 is positive.

Proof: We consider the following cases.

Case (1): $\lambda_1 = \lambda_2 = 0$

Here the structure equations are

$$d\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix}=\begin{pmatrix}\lambda_3+\phi_1 & 0 & 0\\ 0 & \lambda_3-\phi_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\omega^2\wedge\omega^3\\ \omega^3\wedge\omega^1\\ \omega^1\wedge\omega^2\end{pmatrix}$$

and so for the algebra **g** of M is determined by the signs of the diagonal entries of the matrix of the structure constants

$$C=\begin{pmatrix}\lambda_3+\phi_1 & 0 & 0\\ 0 & \lambda_3-\phi_1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

(i) If $\lambda_3 > \phi_1$, so that the diagonal entries are all positive, then **g** = SO(3). Here the manifold is SO(3) with the sub-Riemannian structure D = $(\eta^3)^{\perp}$ and $<,>=\frac{1}{\lambda_3-\phi_1}(\eta^1)^2+\frac{1}{\lambda_3+\phi_1}(\eta^2)^2|_D$, where η is the standard coframing satisfying

$$d\begin{pmatrix}\eta^1\\ \eta^2\\ \eta^3\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\eta^2\wedge\eta^3\\ \eta^3\wedge\eta^1\\ \eta^1\wedge\eta^2\end{pmatrix}$$

(ii) If $\lambda_3 = \phi_1$, then **g** is the Lie algebra of E(2), the group of rigid motions of the Euclidean plane . A specific example is given by the sub- Riemannian structure on R^2 x S^1, with coordinates (x, y, ϕ), induced by the coframing

$$\eta^1=\sqrt{2\phi_1}\ (\cos\phi\, dx+\sin\phi\, dy)$$

$$\eta^2=-d\phi\sqrt{2\phi_1}$$

$$\eta^3=\sin\phi\, dx-\cos\phi\, dy$$

Case (2): $\lambda_2 = 0$ and $\lambda_3 = \phi_1$.

In this case, the structure equations are

$$d\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\end{pmatrix} = \begin{pmatrix}2\phi_1 & 0 & -\lambda_1\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\omega^2\wedge\omega^3\\ \omega^3\wedge\omega^1\\ \omega^1\wedge\omega^2\end{pmatrix}$$

The Lie algebra is determined by the sign of the determinant $2\phi_1 - \frac{\lambda_1^2}{4}$ of a certain 2 x 2 block of the symmetric matrix $\frac{1}{2(C+C^T)}$.

(i) If $2\phi_1 - \frac{\lambda_1^2}{4} > 0$, then it is easily verified that this case is given by the sub- Riemannian structure on $\mathbb{R}^3$ induced by the coframing

$$\eta^1 = e^{-\frac{\lambda_1}{2}z}\left(\left(-\sigma\, sin\sigma z - \frac{\lambda_1}{2}cos\sigma z\right)dx + \left(\sigma\, cos\sigma z - \frac{\lambda_1}{2}sin\sigma z\right)dy\right)$$

$$\eta^2 = -\,dz$$

$$\eta^3 = e^{-\frac{\lambda_1}{2}z}(cos\sigma z\, dx + sin\sigma z\, dy),$$

where $\sigma = \sqrt{2\phi_1 - \frac{\lambda_1^2}{4}}$

(ii) If $2\phi_1 - \frac{\lambda_1^2}{4} = 0$, then it is easily verified that this case is given by the sub – Riemannian structure on $\mathbb{R}^3$.

Case (3): $\lambda_1 = 0$ and $\lambda_3 = -\phi_1$.

It is easy to prove, hence the proof is omitted.

REFERENCES:

[1] **Agrache A. A.,** Exponential mappings for contact sub-riemannian structures, J. Dynamical Systems and Control, 2(3), pp321 – 358.

[2] **Brockett R.,** 1981, Control theory and singular Riemannian geometry, New Directions in Applied Mathematics (Hilton and Young eds.) Springer Berlin.

[3] **Chern S. S. and Hamilton R**. 1985, On Riemannian metrics adapted to three-dimensional contact manifolds, Lecture Notes in Mathematics 1111.

[4] **McDuff D. and. Salamon**, 1998, Introduction to Sympletic topology, Oxford University Press.

[5] **Montgomery R**., 2002, A tour of Sub-Riemannian geometries- their geodesics and applications, Surveys and Monographs, AMS Providence .

[6] **Sternberg S.,** 1964, Lectures on Differential Geometry, Prentice Hall Publications, 400p.

[7] **Tonaka N**., 1975, A differential – geometric study of strongly pseudoconvex manifolds, Lectures in Math, Vol. 9, Kyoto University Press.

[8] **Tanno, S**., 1989, Variational problems on contact Riemannian manifolds. Transactions of the AMS, Vol. 314 (1), pp 349 – 379.

[9] **Webstar S**., 1978, Pseudohermitian structures on areal hyper surface, J. Diff. Geom. 13, pp 25-31.